Radiative Line Transfer and Detailed Balance

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Concept Map

• Line Radiative Transfer
  • Einstein Coefficients
  • Radiative Transfer and Detailed Balance

• Masers
  • Applicability to observational astronomy
CAUTION!
There are a lot of abstract concepts in this section of ERA and we don’t have enough time to go into minute detail. I highly recommend reading through the material on your own (as you should for every topic).
Line Radiative Transfer: Einstein Coefficients

• 3 types of emission:
  • Spontaneous emission ($A_{UL}$)
  • Absorption ($B_{LU}$)
  • Stimulated emission ($B_{UL}$)
    • Negative absorption

• For our model (two-level system):
  \[ E = E_U - E_L \]
  
  • Which leads to a photon:
  \[ \nu_0 = \frac{E}{h} \]
Line Radiative Transfer: Einstein Coefficients

- As outlined in the recombination lines lecture, there is an intrinsic line width to spectral lines.
- For absorption, the coefficient is dependent on the incident radiation field.
- We define the profile-weighted mean radiation energy density:

\[
\bar{u} \equiv \int_0^\infty u_\nu(\nu) \phi(\nu) \, d\nu \quad \rightarrow \quad B_{LU} \bar{u} \quad \& \quad B_{UL} \bar{u}
\]
Line Radiative Transfer: Einstein Coefficients

• In thermodynamic equilibrium (TE) we have stationary states.
• Average rate of emission of photons must balance the average rate of absorption of photons from the radiation field.

\[ n_U A_{UL} + n_U B_{UL} \bar{u} = n_L B_{LU} \bar{u} \]

• Solving for \( \bar{u} \) connects the properties of the quantum system to the blackbody radiation field:

\[ \bar{u} = \frac{A_{UL}}{(n_L/n_U) B_{LU} - B_{UL}} \]
Line Radiative Transfer: Einstein Coefficients

• The Boltzmann equation gives

\[ \frac{n_U}{n_L} = \frac{g_U}{g_L} \exp \left( -\frac{h \nu_0}{kT} \right) \]

where \( g_U \) and \( g_L \) are called statistical weights.

• Combining this with \( \bar{u} \) from the previous slide:

\[ \bar{u} = A_{UL} \left[ \frac{g_L}{g_U} \exp \left( \frac{h \nu_0}{kT} \right) B_{LU} - B_{UL} \right]^{-1} \]
Line Radiative Transfer: Einstein Coefficients

• When we use the Planck radiation law (2.86) for $B_{\nu}(T)$ near $\nu = \nu_0$:

$$\bar{u} \approx \frac{4\pi}{c} \frac{2h\nu_0^3}{c^2} \left[ \exp \left( \frac{h\nu_0}{kT} \right) - 1 \right]^{-1}$$

• This must agree with our equation for $\bar{u}$ from the previous slide for all temperatures $T$.

• Which leads to...
Line Radiative Transfer: Equations of Detailed Balance

• The equations of detailed balance:

\[
\frac{g_L}{g_U} \frac{B_{LU}}{B_{UL}} = 1 \\
\frac{A_{UL}}{B_{UL}} = \frac{8\pi h\nu_0^3}{c^3}
\]

• Some takeaways:
  • If we know one of the three coefficients, we can determine the other two.
  • \(B_{LU}\) cannot be zero; stimulated emission must occur.
Radiative Transfer

• We go back to Chapter 2 for the equation of radiative transfer:

\[
\frac{dI_\nu}{ds} = -\kappa I_\nu + j_\nu
\]

• For pure absorption:

\[
\frac{dI_\nu}{ds} = -\left(\frac{h\nu_0}{c}\right) n_L B_{LU} \phi(\nu) I_\nu
\]

• For stimulated emission:

\[
\frac{dI_\nu}{ds} = \left(\frac{h\nu_0}{c}\right) n_U B_{UL} \phi(\nu) I_\nu
\]
Radiative Transfer

• Adding these two yields the **net absorption coefficient**:

\[
\kappa = \left( \frac{h\nu_0}{c} \right) \left( n_L B_{LU} - n_U B_{UL} \right) \phi(\nu)
\]

• For spontaneous emission:

\[
\frac{dI_\nu}{ds} = j_\nu = \left( \frac{h\nu_0}{4\pi} \right) n_U A_{UL} \phi(\nu)
\]

• Now we can write the full spectral-line equation of radiative transfer:

\[
\frac{dI_\nu}{ds} = - \left( \frac{h\nu_0}{c} \right) \left( n_L B_{LU} - n_U B_{UL} \right) \phi(\nu)I_\nu + \left( \frac{h\nu_0}{4\pi} \right) n_U A_{UL} \phi(\nu)I_\nu
\]
Radiative Transfer

• Interestingly, we can use these equations to eliminate $A_{UL}$, $B_{UL}$, and $B_{LU}$ and Kirchhoff’s law in LTE to recover the Boltzmann equation.

• Our derivations are not specific to total TE, but also LTE!

• Using this, the **net opacity coefficient** in LTE is:

\[
\kappa(\nu) = \frac{c^2}{8\pi \nu_0 g_L} n_L A_{UL} \left[ 1 - \exp \left( -\frac{h\nu_0}{kT} \right) \right] \phi(\nu)
\]
Radiative Transfer

• Remember, at radio frequencies $h \nu / kT << 1$ leading to stimulated emission nearly cancelling pure absorption and significantly reducing line opacity.

• Also, as $\kappa \propto 1/T$ and $B_\nu \propto T$ the product $\kappa B_\nu$ is independent of line temperature.

• The brightness of an optically thin radio emission line is proportional to the column density of emitting gas but can be nearly independent of the gas temperature.

  • Question: If we observe the HI line flux of an optically thin galaxy, what can we interpret from it?
Excitation Temperature

• When our two-level system is not in LTE, its excitation temperature is defined by:

\[ \frac{n_U}{n_L} \equiv \frac{g_U}{g_L} \exp \left( -\frac{h\nu_0}{kT_x} \right) \]

• \( T_x \) causes collisional excitations and de-excitations.
• We need to revise our detailed balance equation:

\[ n_U(A_{UL} + B_{UL}\bar{u} + C_{UL}) = n_L(B_{LU}\bar{u} + C_{LU}) \]
Masers

• If the upper energy level is overpopulated \( \left( \frac{n_U}{n_L} > \frac{g_U}{g_L} \right) \) then \( T_x \) is negative and the net line opacity is negative

• Huh???

• The source is actually brighter due to the medium.

• This is called maser (microwave amplification by stimulated emission of radiation) amplification and it is very common at radio frequencies.

• Can have line brightness temperatures as high as \( 10^{15} \) K!
Masers

• Assume \( g_U = g_L \).

• We will also assume the line profile is a Gaussian with FWHM \( \Delta \nu \) and (from last lecture) we can use the numerical approximation \( \phi(\nu_0) \approx 1/\Delta \nu \).

• Then:

\[
\tau = \frac{h\nu_0 B}{c\Delta \nu} \int (n_U - n_L) ds
\]

is called the **maser gain** and amplifies the signal by a factor \( \exp(|\tau|) \).

• \( s > 10^{13} \text{ cm} \) (about \( 10^{-5} \text{ pc}, 1 \text{ AU} \)) for significant gain to occur.
Masers

• Masers are like lasers – they need to be “pumped” by an energy source or the upper energy levels are depopulated quickly.

• The maser is saturated if the stimulated emission rate is limited by the pump luminosity; it is unsaturated if the pump is more than adequate.

• Where is a good place to look for masers?
Masers

• Supermassive black holes!
• Masers are great sources for measuring astrometry (motions) and thus are a great tool for measuring black hole mass.
\[ V_z = \sqrt{\frac{GM}{R}} \sin \theta \]

\[ V_z = \sqrt{\frac{GM}{R}} \]

\[ V_z = \sqrt{\frac{GM}{R}} \frac{b}{R} = \sqrt{\frac{GM}{R^3}} b \]
Masers: Question for you

• Using the information from the figure, what is the mass of the object causing the Keplerian orbits of these masers? Find the mass density of this object. Compare it to a star cluster of mass 1000 solar masses and radius 10 pc.
  • $V_{\text{rot}} = 900 \text{ km/s}$
  • $V_{\text{galaxy}} = 450 \text{ km/s}$
  • $R = 0.1 \text{ pc}$
  • $V = \sqrt{GM/R}$
Masers

• There is an ongoing project called the Megamaser Cosmology Project which uses masers to measure all sorts of extragalactic parameters for many systems.