Radio Astronomy (ASTR700) Problem Set #4
DUE 25 Feb, but get started early!!!
Review, Filled-Aperture Telescopes, and a bit of Radiometers

Each question part is worth 10 points unless otherwise stated [total: 70/70].

1. A few review questions.

(a) Coherent (maser) radiation from a source at a distance of 2.3 kpc has a flux density of $10^3$ Jy over a frequency band of 1 kHz. If it is isotropic, what is the power radiated?

Answer: For an isotropic emitter we know that the total power radiated can be written as

$$L = 4\pi D^2 S$$

So all we need to do is convert our distance to meters ($7 \times 10^{19}$ m) and understand that the total flux density $S = S_\nu \Delta \nu = 10^3$ Jy $\times$ $10^3$ Hz $\times$ $10^{-26}$ W/Hz/m$^2$ = $10^{-20}$ W/m$^2$. Then

$$L = 4\pi (6.9 \times 10^{19})^2 10^{-20} = 6 \times 10^{20} \text{ W}$$

(b) Pluto is 38.5 AU from the Earth at closest approach. Its radius is thought to be $\sim 1200$ km. At 1.3 mm, the measured flux density of Pluto is 50 mJy. Calculate Pluto’s temperature.

Answer: I’m going to work symbolically here and recognize that we want the temperature, and the following things are true:

$$T = \frac{c^2 S_\nu}{2k\nu^2 \Omega}$$

is the brightness temperature in terms of source flux and solid angle subtended (you showed this on HW#1) and

$$\Omega = \Omega_{\text{pluto}} \simeq \pi \left( \frac{R_{\text{pluto}}}{D_{\text{pluto}}} \right)^2 \simeq 1.4 \times 10^{-13} \text{ sr}$$

and $\lambda \nu = c$ and therefore we can combine these expressions to write

$$T = \frac{\lambda^2 S_\nu D_{\text{pluto}}^2}{2\pi k R_{\text{pluto}}^2}$$

Plugging in our knowns, we arrive at around 224 K.
(c) Considering Pluto again, for a 30 m radio telescope, calculate the antenna temperature. For this problem, assume that the telescope has a tapered aperture illumination as in ERA section 3.2.5 and aperture efficiency of 100%.

**Answer:** For ERA section 3.2.5 the only relevant difference was that our FWHM beam size is given as:

$$\theta = 1.2 \frac{\lambda}{D}$$

thus

$$\Omega = \frac{\pi}{4} \theta^2 = \frac{\pi}{4} \left(1.2 \frac{\lambda}{D}\right)^2 = 2.1 \times 10^{-9} \text{sr}$$

(7)

Where \(D\) is dish diameter. For such a telescope, we want to calculate the antenna temperature that Pluto would give us. We know that for an unresolved source,

$$\frac{T_A}{T_B} = \frac{\Omega_s}{\Omega_A}$$

(8)

and we must use this expression since Pluto will be unresolved in our beam. Note that the original problem had a typo for aperture efficiency; it was intended to be 100% efficiency, not 70% (both were listed; fine if you used 70%). So to get the antenna temperature we consider the brightness temp of pluto (given above since it’s opaque; \(T_B = 224 \text{K}\)) and the solid angle of pluto given in part (b)’s solution above, in addition to the solid angle of the antenna in question, given above. Thus we find

$$T_A = 0.015 \text{K}$$

(9)

Note: if you used anything not 100% efficiency it would scale this value directly by \(\eta\) (not as many photons actually contribute to antenna temp in that case!).

(d) Calculate the conversion factor from Jy to K for a given antenna temperature for a single VLA (dish size 25 m) if the aperture efficiency is 0.5.

**Answer:** We know that

$$P = kT = \frac{SA_e}{2}$$

(10)

or reforming this in terms of the conversion factor we need and using the fact that \(A_e = \eta A_g\) where \(A_g\) is the geometric area of the aperture:

$$\frac{T}{S} = \frac{\eta A_g}{2k}$$

(11)

This is the conversion factor we’re looking for (multiply a Jansky value by this factor to get Kelvin); no worries if you wrote the inverse of this, since the question wasn’t explicit). Plugging in the knowns I get:

$$\frac{T}{S} = \frac{0.5 \pi (25/2)^2}{2(1.38 \times 10^{-23})} = 9 \times 10^{24}$$

(12)
Note that we did this in SI units (thus it’s in $K/(W/Hz/m^2)$), so we need to fix this to be in $K/Jy$ by multiplying by $10^{-26}$. Doing this, we obtain

$$\frac{T}{S} = 9 \times 10^{-2} K/Jy$$  \hspace{1cm} (13)

(e) The Arecibo telescope has a diameter of 300m. Given the values from your previous homework’s question 5, what is the aperture efficiency of Arecibo’s 2.3 GHz feed? [Note, HW3 is on the course site if you need it.]

\textbf{Answer:} In the previous homework we learned that Arecibo’s effective area was $2.5 \times 10^8 \text{ cm}^2 = 2.5 \times 10^4 \text{ m}^2$. The aperture efficiency is given by

$$\eta = \frac{A_e}{A_g}$$  \hspace{1cm} (14)

So plugging in for $A_g = \pi R^2 = \pi (150 \text{ m})^2$, we get $\eta = 35\%$.

(f) We mostly use paraboloid dishes to improve the directivity of our gain. Compare a simple dipole to an aperture with tapered illumination by determining what fraction of the whole sky is covered by the beam solid angle of each. The beam solid angle of a tapered illumination is discussed in ERA section 3.3.3, but the dipole you will have to consider how to determine.

\textbf{Answer:} Using the definition of beam solid angle

$$\Omega_A = \frac{4\pi}{G_0}$$  \hspace{1cm} (15)

we can see that we can solve this problem by computing the beam solid angle of a dipole and then comparing that to the total solid angle in the sky ($4\pi$), or we could recognize that the quantity $1/G_0$ actually represents the fraction of the sky covered by the beam:

$$\frac{1}{G_0} = \frac{\Omega_A}{4\pi}$$  \hspace{1cm} (16)

... which is the quantity we’re trying to compare! Very cool.

Using the above, for a dipole in class we computed that $G_0 = 3/2$. So, a dipole’s beam solid angle covers 2/3 of the sky. That is a lot of the sky! The dipole does not have much directivity.

For a tapered illumination of a dish we see in ERA 3.3.3 that $\Omega_A$ is given by

$$\Omega_A \simeq 1.1 \theta^2_{\text{HPBW}} \simeq 1.1 \frac{\lambda^2}{D^2}$$  \hspace{1cm} (17)

For dish-based radio observations, it will typically be the case that $\lambda < D$; thus, $\Omega_A$ will be exceedingly small. For bigger dishes, the beam solid angle gets even...
smaller. Let’s plug in typical values just to compare... \( \lambda = 20 \text{ cm} \) and \( D = 50 \text{ m} \), so \( \theta_{\text{HPBW}} = 0.004 \text{ sr} \). Then \( \Omega_A \simeq 2 \times 10^{-5} \), which is about \( 10^{-6} \) of the whole sky! The dish has a very high directivity for its power transmission pattern.

2. At some observing frequency, the sky density (number per steradian) of radio-loud blazars at a flux density higher than \( S_J \) is \( N = 300 S_J^{-3/2} \text{ sr}^{-1} \), where \( S_J \) is flux density in units of Jansky. Assume the sky only has radio-loud blazars. If we define our “confusion limit,” \( \sigma_c \), to be the flux density at which we can no longer resolve two distinct blazars, write an expression for the confusion limit if you’re observing at a frequency of \( \nu \) with a dish of diameter \( D \).

**Answer:** We stipulate in the problem that we will be “confused” if there are more than two sources in our beam, on average. So in other words we need to ensure that something like this is true:

\[
N \leq \frac{2}{\Omega} \tag{18}
\]

where \( \Omega \) represents the solid angle of our telescope’s beam. Testing it for \( N \leq 1/\Omega \) would also make sense (and some of you did this). We know that

\[
\Omega = \pi \left( \frac{\theta_{\text{HPBW}}}{2} \right)^2 = \pi \left( \frac{c}{2 \nu D} \right)^2. \tag{19}
\]

Plugging in the above expression and rewriting \( N \) with the expression given in the equation, we then can solve for the limiting flux density:

\[
S \geq \left( \frac{2 \nu^2 D^2}{75 \pi c^2} \right)^{-2/3} \tag{20}
\]

(or whatever form you want to express this; it’s right as long as it can be altered to look like this expression!).

3. In class, we calculated the point source response \( P(\theta) \) for a canonical filled-aperture antenna with perfect illumination. We also learned that if a very bright source sits near a sufficiently high sidelobe, it could potentially “leak” false signal into your target of interest. ERA section 3.2.5 computes the power pattern for a more standard aperture with a tapered illumination pattern. Let’s say you’re observing with Parkes telescope \((D = 64 \text{ m})\) at a frequency of 1 GHz. **For this illumination pattern, at what minimum angle must you keep the sun away from the telescope’s pointing direction to ensure that the Sun’s power will be attenuated by at least 30 dB?** Assume only the lobe structure coming from the primary aperture, i.e. there are no other sidelobes from feed legs or other telescope sub-structures. Simplify your life by also assuming that the sun is a point source, i.e. you can assume all its flux is focused at a single-valued \( \theta \). Note, if you can’t or don’t want to solve this analytically...
try writing a short computer program to solve it numerically (e.g. a for-loop solving the power level over a reasonable range of values). It might help guide your code to plot the power pattern. If you do that, please attach a print-out of your program to your homework!

Answer: I wrote a Perl script to solve this problem. I did a few checks within the script to make sure I was getting it right, and you may have solved the problem differently. Here is a screenshot of my script.

```perl
#!/usr/bin/perl

# Define constants/knowns
$nu = 1.0e9; # Obs frequency in GHz
$D = 64;  # Dish diameter, meters
$c = 3e8; # Speed of light (m/s)
$pi = 3.14159;

# Calculate wavelength and beam size
$lambda = $c/$nu; # Wavelength, meters
$hpbw = $lambda/$D; # Half-power beam width in radians

# Sanity check: Print what we expect the half-power beamwidth to be. The first sidelobe should be a factor of a few beyond the half-power beamwidth.
printf STDERR "Parke half-power beamwidth is %.2f arcminutes (%f sr)\n", $hpbw*180/$pi*60, $hpbw;

# Loop over the power pattern from 0 out to a few sidelobes.
$below = 0; # This is a switch that tells me whether I'm below -30dB or not.
for($i=0;$i<0.05;$i+=0.0001){
    # This equation is given by ERA eq. 3.94
    $P =
    (cos($pi*$i*$D/($lambda))
    / (1 - 4*$i*$D/($lambda))**2)
    **2;

    # Convert P to decibels
    $P = 10*log10($P);

    # Notify if we cross the -30dB threshold.
    if (!$below && $P <=-30){
        $below = 3;
        printf STDERR "Dropped below -30 at $i sr (%.2f arcmin)\n", $i*180/$pi*60;
    } elsif ($below && $P >=-30){
        $below = 0;
        printf STDERR "Went back above -30 at $i sr (%.2f arcmin)\n", $i*180/$pi*60;
    }
}

printf "$i\t$P\n";
}

# Perl doesn't have a built in log10 function so I wrote my own
sub log10{
    $val = shift @_;#
    return(log($val)/log(10));
}
```

To have my own sanity check that I was getting things roughly right I calculated the half-power beam width, and had a for loop that printed out all the values scanning
out through the beam over small steps in angle. The half-power beamwidth helped me understand if what I was looking at was what I thought it should look like (the main lobe should be about as big as the HPBW angle)! Here is the plot I made from

At least in form this looks somewhat like the figures in ERA did, so I know I'm more or less doing things right (the x scale is different because ERA plots normalized values, whereas we're using a real telescope). While I could have just looked at the list of values and checked where it finally went beyond the -30dB point and stayed there, just to make things fancy I had my script print a message to the terminal whenever it passed the -30dB mark. At some point, it crossed under and stayed there. At that point, I knew that no matter where the sun was in the sidelobe pattern, it would always be attenuated by at least 30dB.

As my program notes on the last line, this appears to be at about 0.02 sr, or about 74 arcminutes, away from the same beam. So in principle we can point within a degree of the solar disk and the sun’s signal will still be greatly attenuated! Radio astronomy is amazing.

4. Note: This problem aims to help you consider the type of analysis you might do when
proposing for observations. It relies on concepts discussed in the Radiometers lecture, Tuesday 18 Feb. As explored on HW1 and discussed in class, the confusion limit is the flux level at which a telescope can no longer resolve discrete sources because the sky density is too high. You wish to study a highly compact ($\ll 1$ arcsec) source at a distance of 500 Mpc whose luminosity is predicted to be $L_{1\text{GHz}} = 5 \times 10^{21}$ W Hz$^{-1}$. Would this observation be feasible with a single dish telescope with the parameters given below? Assume you want to obtain a high-confidence detection, with at least a signal-to-noise of 10. Consider in turn the:

(a) Telescope sensitivity.

(b) Telescope resolution and confusion limit (ERA 3.6.3).

These parameters are on par with front-end receiving systems at Green Bank and Parkes telescopes:

- Telescope has 0.70 efficiency.
- The system noise temperature is 30 K (includes all contributions in ERA Eq. 3.150).
- System bandwidth is 400 MHz, with a central frequency of 1200 MHz.
- System receiver includes two orthogonal dipoles; you will record each orthogonal polarization for analysis. Assume your target is unpolarized.
- You can get up to 2 hours of on-target observing time.

Answer: Note, I solved this in terms of answering whether the factors would give you a feasibly-sized dish (or basically, how big a dish would I need to get enough sensitivity?). As we noted in class, some of you took a very different (and in most cases correct) approach to this, solving to understand how big the radiometer noise $\sigma_J$ and the confusion noise $\sigma_c$ were for a standard dish like GBT or Parkes and judging the feasibility that way. That is all fine as long as your logic was sound and your equation applications were right! There is almost always more than one way to do approach a science question.

The diameter of the telescope, $D$, is not given (nor is the effective area or gain), so let’s focus on that as a test of feasibility. There are two limitations you need to consider here: the resolution limit of the telescope (i.e. can you resolve the object from the source background?) and the sensitivity limit of the telescope (will the system’s sensitivity enable you to reach a low enough limit?). Let’s consider these in turn, but first we need to find the flux density of the object itself. Based on its luminosity and distance, the source flux is

$$S_\nu = \frac{L_\nu}{4\pi d^2} = \frac{(5 \times 10^{21} \text{ W/Hz})(10^{-26} \text{ Jy}/(\text{W/Hz/m}^2))}{4\pi(500 \times 10^6 \text{ pc})(3 \times 10^{16} \text{ m/pc})} \approx 167 \mu\text{Jy}$$

And we want a measurement at 10 times the noise, so we need to reach a noise floor of 16.7 $\mu$Jy. Note, we will assume an average wavelength of $c/1200$ MHz $\approx 25$ cm.
Sensitivity limit of telescope

Using the radiometer equation

\[ \sigma_T = \frac{T_{\text{sys}}}{\sqrt{n_p B t}} = \frac{30 \text{ K}}{\sqrt{(2)(400 \times 10^6 \text{ Hz})(7200 \text{ s})}} = 1.25 \times 10^{-5} \text{ K} \]

This sensitivity is given in Kelvins, whereas we want to know our sensitivity limit in Janskies so we can make sure it is higher than 16.7 \( \mu \)Jy. For this we need to scale the temperature variations by the “degrees per flux unit,” as seen a few times already and formally introduced in the Radiometers lecture. We can “convert” \( \sigma_T \) to a flux variance \( \sigma_S \) by altering it by this scaling factor. We can re-form the definition of antenna temperature to find that

\[ \frac{S}{T_A} = \frac{\sigma_S}{\sigma_T} = \frac{2k}{A_e} \]

where \( k \) is the Boltzmann constant. Finally, our aperture efficiency is

\[ \eta_A = 0.7 = \frac{A_e}{A_{\text{geom}}} \rightarrow A_e = \eta_A \left( \frac{\pi D^2}{4} \right) \]

therefore we can at last relate this back to the size/diameter of the dish, which must be larger than

\[ D > \left( \frac{\sigma_T}{\sigma_S} \cdot \frac{8k}{\pi \eta_A} \right)^{1/2} = \left( \frac{(1.25 \times 10^{-5})(8)(1.38 \times 10^{-23})}{\pi (16.7 \times 10^{-6} \times 10^{-26})(0.7)} \right)^{1/2} \approx 61 \text{ m.} \]

Note that officially, the “system equivalent flux density” of a dish, SEFD, is defined as

\[ \text{SEFD} = \eta \frac{2kT_{\text{sys}}}{A_e} \]

and hence the radiometer equation can be rewritten as

\[ \sigma_s = \frac{\text{SEFD}}{\sqrt{n_p B t}} \]

to reflect the expected RMS variations in a flux density.

Resolution limit of telescope

Telescope resolution is given by

\[ \theta_{\text{HPBW}} \approx \frac{\lambda}{D} \]

Where D is the diameter of the dish. The confusion limit can be considered as a noise floor, so we want to make the dish big enough so that the confusion noise is \( \leq 16.7 \mu \)Jy. A formulation for confusion noise is given in ERA section 3.6.3 (equation 3.163 in my version of the book). Note that this formulation relies on the idea that there is some number density of sources on the sky that is a function of both flux and frequency.
The confusion limit occurs when the FWHM can no longer (on average) resolve two point sources—that is, there are so many sources that there is more than one per beam on average. Using the formulation from ERA and noting that $\theta_{\text{FWHM}} = \theta_{\text{HPBW}}$, I set $\nu = 1.2$ (units of GHz) and $\sigma_c = 0.0167$, and input 0.17 arcmin to the top equation, which gave me $\sim 5\,\mu\text{Jy}$, indicating that the FWHM would be larger than 0.17 arcmin so the top equation should apply. Thus

$$
\left( \frac{\theta_{\text{FWHM}}}{\text{arcmin}} \right)^2 = \left( \frac{1}{0.2} \right) \left( \frac{\sigma_c}{\text{mJy beam}^{-1}} \right) \left( \frac{\nu}{\text{GHz}} \right)^{0.7}
$$

and $\theta_{\text{HPBW}} \leq 0.092$ arcmin $= 4.6$ arcsec. Thus leading to the constraint

$$
D > \frac{0.25\,\text{m}}{2.2 \times 10^{-5}\,\text{rad}} \approx 1.1 \times 10^4\,\text{m} = 11\,\text{km}.
$$

Ouch! This is one big dish diameter. So, the ability for a single dish to observe this source is completely dominated by the confusion limit encountered by single dishes. By no coincidence, we will soon be talking about interferometers, which side-step this problem in an elegant way.