Radio Astronomy (ASTR700) Problem Set #1
Quantifying light, radiative transfer, and the Rayleigh-Jeans limit

To ensure you obtain full credit, explain all of your work; don’t just write equations and numbers. Tell me (briefly) what you’re doing and why. Try all parts of a problem, even if you can’t solve earlier parts.

Each question part is worth 10 points unless otherwise stated [total: 80/80].

1. Because radio telescopes are of variable resolution, a common unit of measure for intensity that you may see in your research is Jy/beam\(^{-1}\), where a “beam” is a resolution element on the sky. A unit commonly used in presenting data from infrared satellites is MJy sr\(^{-1}\). If the intensity is uniformly distributed, convert 1 MJy sr\(^{-1}\) to Jy beam\(^{-1}\) for a 1 arcmin diameter beam.

**Answer:** “Beam” represents a solid angle. 1 arcminute is small enough to obey the small angle approximation, such that the solid angle subtended by beam diameter \(\phi = 1\) arcmin is:

\[
\Omega_{\text{beam}} = \pi \theta^2 = \pi \left(\frac{1'}{2}\right)^2 = \pi \left[\left(\frac{1'}{2}\right) \left(\frac{\pi \text{ rad}}{180^\circ}\right) \left(\frac{1^\circ}{60'}\right)\right]^2 = 6.6 \times 10^{-8} \text{ sr}
\]

(1)

And therefore

\[
\frac{1 \text{ MJy}}{\text{sr}} = \frac{10^6 \text{ Jy}}{\text{sr}} \left(10^8 \text{ sr beam}^{-1}\right) = 6.6 \times 10^{-2} \text{ Jy beam}^{-1}
\]

(2)

2. Anti-collision radars have been proposed to be installed on cars and to operate at 70 GHz. The bandwidth would be 100 MHz, and at a distance of 3 m, the power per unit area is \(10^9 \text{ W m}^{-2}\). Assuming uniform power over the proposed band, compute the flux density of the radar that would be observed at a distance of 1 km. At what distance would such transmitters disturb radio telescope observations at the mJy level?

**Answer:** The power per unit area is \(10^9 \text{ W m}^{-2}\); to make this a flux density let’s distribute it evenly over the proposed 100 MHz bandwidth such that at a distance of 3 meters:

\[
S_{\nu,3m} = \frac{10^9 \text{ W/m}^2}{100 \times 10^6 \text{ Hz}} = 10 \text{ W/m}^2/\text{Hz} = 10^{27} \text{ Jy}
\]

(3)

But we’re asked what the flux density at a distance of 1 km would be. We know that flux is proportional to distance as \(D^{-2}\) such that we can write a relation

\[
\frac{S_{\nu,3m}}{S_{\nu,D}} = \left(\frac{3 \text{ m}}{D}\right)^{-2}
\]

(4)
Plugging in our numbers and solving for flux density at distance $D$, we get the relation:

$$S_{\nu,D} = \frac{9 \times 10^{27}\ \text{Jy m}^2}{D^2}$$ (5)

So for a distance of 1 km, $S_{\nu} = 9 \times 10^{-5}\ \text{W/m}^2/\text{Hz} = 9 \times 10^{21}\ \text{Jy}$. For a flux of 1 mJy, if the radio telescope is unshielded the transmitter would have to stay $D = 3 \times 10^{15}\ \text{m}$ away! Seems a little impractical.

3. Let’s explore the Rayleigh-Jeans limit for blackbody emission, which is a relation that by now I hope you would be able to state clearly if I woke you suddenly at 3am.

   (a) (5 points) Determine the expression for brightness temperature $T_B$ in terms of observing frequency $\nu$ and the target’s directly measurable source quantities $S_\nu$ and $\Omega$.

   Answer: We know that:

   $$T_B = \frac{I_\nu c^2}{2k\nu^2} \quad \text{and} \quad S_\nu = I_\nu \Omega$$ (6)

   where the latter is true if the emission you are looking at is evenly distributed, which for simplicity we will assume here (you might also derive this in terms of the full expression for $S_\nu$, if desired). Plugging this straight in we find:

   $$T_B = \frac{S_\nu c^2}{2k\nu^2 \Omega}$$ (7)

   (b) (5 points) In class, we saw the Rayleigh-Jeans approximation and intensity in terms of unit frequency $d\nu$. Using Eq. 2.5 of ERA, and starting with our frequency-based form of the Rayleigh-Jeans approximation, derive an expression for $B_\lambda(T,\lambda)$ such that the blackbody intensity $B_\lambda$ is instead in terms of unit wavelength.

   Answer: The RJ approximation to the Planck function is:

   $$B_\nu(T) \simeq \frac{2kT\nu^2}{c^2}$$ (8)

   Using ERA 2.5 we know that:

   $$\nu B_\nu = \lambda B_\lambda \quad \text{and separately we know} \quad c = \lambda \nu$$ (9)

   Combining the above equations we find that

   $$B_\lambda(T) = \frac{\nu}{\lambda} B_\nu = \frac{2kT\nu^3}{\lambda c^2} = \frac{2kT}{\lambda^3} \frac{c^3}{\lambda^4} = \frac{2kTc}{\lambda^4}$$ (10)
(c) Given a blackbody spectrum, derive a relation in the Rayleigh-Jeans limit for
the photon flux (number of photons emitted per m$^2$ per second per Hertz) as a
function of frequency. It would be pertinent for you to recall that a blackbody is
an isotropic emitter (that is, it emits equally in all directions).

Answer: We want to know the emitted photon flux as a function of frequency. Black-
bodies are isotropic, so the flux is emitted over a solid angle of 4π. Brightness, B, can
be integrated over total solid angle to determine the total energy [per units] coming
out of the source. We then have the net energy for the blackbody, and to determine
the number of photons vs ν, we can divide by photon energy $E_\nu = h\nu$. Putting all this
together:

$$\frac{#}{m^2, Hz s} = \frac{4\pi B}{E_\nu} = \frac{8\pi kT\nu}{hc^2}$$

(11)

Where I have in the above put in the RJ relation for $B$, as required by the problem.

4. At a frequency of $\sim$20 GHz, the Green Bank Telescope can confidently detect flux
densities down to around 0.1 mJy. Below this, the telescope begins to run into its
“confusion limit,” where there are too many objects at the same flux density, such
that no single object can be identified.$^1$

(a) What is the maximum distance we could detect thermal emission from a neutron
star ($T \sim 10^5$ K, radius $R \sim 10$ km) given this confusion limit?

(b) (5 points) The nearest neutron star is around 80 pc away, and the most sensitive
radio telescopes can detect down to around 1 $\mu$Jy. Comment: could humans ever
detect thermal emission from neutron stars? Consider what astronomers might try
to do to have a better chance at detecting thermal emission from a pulsar (note,
this question might be easier to answer after our blackbody radiation lecture).

Answers: We know that for small angles the flux $S_\nu = I_\nu \Omega = I_\nu \pi R^2 / D^2$, where $D$ is
the distance to the star. Thus a star will be detected if

$$D \leq \sqrt{\frac{I_\nu \pi R^2}{S_{lim}}}; I_\nu = B_\nu = \frac{2kT_\nu^2}{c^2},$$

(12)

the latter we can use to consider thermal (blackbody) emission in the RJ approxima-
tion. Therefore to simplify...

$$D \leq \frac{\nu R}{c} \sqrt{\frac{2\pi kT}{S_{lim}}}$$

(13)

$^1$This doesn’t matter for the problem itself, however in reality GBT’s confusion limit at this frequency is
lower (around 0.04 mJy). Here I have assumed we want to detect the star with a signal-to-noise ratio of $\geq 6$. 
So with GBT’s 20 GHz confusion limit converted to MKS units, \( S_\nu = 2 \times 10^{-30} \text{ WHz}^{-1} \text{ m}^2 \), we can plug these in and get

\[
D \leq \frac{(2.0 \times 10^{10})(10^4)}{3 \times 10^8} \left( \frac{2\pi \times 1.4 \times 10^{-23} \times 10^5}{2 \times 10^{-30}} \right)^{1/2}
\]

(14)

Which gives us \( D \lesssim 10^{12} \text{ m} \simeq 3 \times 10^{-5} \text{ parsecs} \simeq 7 \text{ AU} \). So in response to (b), basically it would have to be in the solar system! We apparently have no pulsars in our solar system, and so would have no chance at detecting even the nearest pulsar’s blackbody emission in the radio band (given \( D \propto \nu S_\nu^{1/2} \) and radio frequencies up to \( \sim 1 \text{ THz} \)). Fortunately, pulsars also emit non-thermal emission, which allows us to detect them up to much greater distances. Going to much higher frequencies can help us (thermal emission from neutron stars has, for instance, been detected at optical/X-ray wavelengths). Note that at those frequencies the Planck relation, not the Rayleigh-Jeans limit as used above, applies. You could also use Wien’s Law to determine the peak frequency/wavelength of the emission at the given pulsar temperature, which is in the optical band.

5. Opacity and Kirchhoff’s Law:

(a) Starting from the equation of radiative transfer, assuming the Rayleigh Jeans limit for blackbody radiation, use Kirchhoff’s law and the definitions of optical depth \( \tau \), brightness temperature \( T_B \) and blackbody brightness \( B_\nu \) to show that

\[
\frac{dT_B}{d\tau} = T_B - T
\]

where \( T \) is the temperature of a radiative source.

(b) In class (rewritten here in the notation of ERA), in assuming a medium in local thermal equilibrium we were given that

\[
I_\nu(\tau) = I_\nu(0)e^{-\tau} + B_\nu(T)(1 - e^{-\tau})
\]

(16)

is a solution to the above equation. Use this expression to find expressions for \( I_\nu(\tau) \) in the optically thin (\( \tau \to 0 \)) and thick (\( \tau \to \infty \)) regimes.

(c) (5 points) Discuss briefly (1-2 sentences) in words your interpretation of the result of part (b).

Answers: We start with the equation for radiative transfer:

\[
\frac{dI_\nu}{ds} = -\kappa_\nu I_\nu + j_\nu
\]

(17)

divide all by \( -\kappa_\nu \):

\[
-\frac{dI_\nu}{\kappa_\nu ds} = I_\nu - \frac{j_\nu}{\kappa_\nu}
\]

(18)
The denominator on the left (including the negative sign out front) represents our definition of $d\tau_\nu$, while on the right we can replace the second term using Kirchoff’s law:

$$\frac{j_\nu}{\kappa_\nu} = B_\nu$$

(19)

to end up with:

$$\frac{dI_\nu}{d\tau_\nu} = I_\nu - B_\nu$$

(20)

Let’s recall how we defined brightness temperature:

$$I_\nu = \frac{2k\nu^2T_B}{c^2}$$

(21)

Putting equation 21 and the RJ approximation for $B_\nu$ into equation 20, we see that the constants $2k\nu^2/c^2$ cancel out on both sides of the equation, leaving the desired result:

$$\frac{dT_B}{d\tau_\nu} = T_B - T$$

(22)

For part (b), for the optically thick regime, as $\tau \rightarrow \infty$, $e^{-\tau} \rightarrow 0$, and so the equation becomes:

$$I_\nu = B_\nu \quad \text{[optically thick regime]}$$

(23)

For the optically thin regime, as $\tau \rightarrow 0$, $e^{-\tau} \rightarrow 1$ and so:

$$I_\nu(\tau) = I_\nu(0) \quad \text{[optically thin regime]}$$

(24)

In other words, in the optically thick regime, any incident emission is absorbed and all you see is a blackbody. In the extreme optically thin regime, any interfering medium is basically transparent and has no effect on the radiation!